Martingales and MCs (13.7 Le Gall)

$f : S \to \mathbb{R}$ is harmonic if $f(x) = \underset{\text{semigroup of operators } \{P^k\}}{Pf(x)} = \sum p(x,y) f(y)$

We can also say f is sub or super if $f \le Pf$ or $f \ge Pf$ respectively.

Le Gall restricts to nonnegative functions.

Prop 1) f is (super) harmonic iff $f(X_n)$ is a (super) martingale under P_x.

2) Let $F \subset S$ and $F^c = S \setminus F$ $\quad T_{F^c}$ = hitting time of F^c (leaving time of F)

If f is (super) harmonic on F then $f(X_{n \wedge T_{F^c}})$ is a (super mart)

Pf: $E_x[f(X_n) \mid \mathcal{F}_{n-1}] = E_{X_{n-1}}[f(X_1)] = \sum p(X_{n-1}, y) f(y) = f(X_{n-1})$ (if harmonic)

$\Rightarrow f(X_n)$ is a martingale.

Going backwards again $\sum p(X_0, y) f(y) = f(X_{n-1})$ is the martingale condition.

For part 2, one has to be a bit careful, but its more or less the same idea as "stopped martingales are martingales".

★ Recall: $T_F = \inf\{n \ge 0 : X_n \in F\}$ (1st hitting time) $\Rightarrow T_{\{x\}} \ne T_x = \inf\{n > 0 : X_n = x\}$ (1st return time)

But $T_{\{x\}} = T_x$ if $X_0 \ne x$. The difference will be important below.

Theorem 13.36 Let $F \subset S$ be a nonempty proper subset. Let $g : S \to \mathbb{R}_+$ be bounded.

1) $h(x) = E_x\left[g(X_{T_{F^c}}) 1_{\{T_{F^c} < \infty\}}\right]$ is harmonic on F

2) If $T_{F^c} < \infty$ a.s. then h is the UNIQUE fn on S that is harmon on F and $h(x) = g(x)$ on F^c (because $T_{F^c} = 0$)

Ex: $h(x) = P_x(T_{F^c} < \infty)$ is harmonic on F and satisfies $h(x) = 1$ on F^c

This is a different theorem. It finds a harmonic fn (and consequently a martingale).

If $x \in F$, then $\omega_0 \notin F^c$ and obviously $X_{T_{F^c}} \circ \theta_1 = X_{T_{F^c}}$ and

$$g(X_{T_{F^c}}) 1_{\{T_{F^c} < \infty\}} = g(X_{T_{F^c}} \circ \theta_1) 1_{\{T_{F^c} \circ \theta_1 < \infty\}} \quad P_x \text{ a.s.}$$

Then $h(x) = E_x\left[g(X_{T_{F^c}}) 1_{\{T_{F^c} < \infty\}}\right] = E\left[g(X_{T_{F^c}} \circ \theta_1) 1_{\{T_{F^c} \circ \theta_1 < \infty\}}\right]$

$$= E_x\left[E_{X_1}\left[g(X_{T_{F^c}}) 1_{\{T_{F^c} < \infty\}}\right]\right] = E_x\left[h(X_1)\right] = Ph(x)$$

$\Rightarrow$ h is harmonic on F.

(ii) Uniqueness. Suppose h' is another bounded nonnegative harmonic fn on F, that coincides with g on F^c

Then $h'(X_{T_{F^c} \wedge n})$ is a martingale

$h'(x) = E_x\left[h'(X_{T_{F^c} \wedge n})\right] \rightarrow E_x\left[h'(X_{T_{F^c}})\right]$ by bounded conv.

$= E_x\left[g(X_{T_{F^c}})\right] = h(x)$

by previous.

The space of positive harmonic fns on a space is called the Martin boundary.
Of course harmonic requires a Markov kernel to be defined as well.

Ex: Let $S = \{0, \cdots, m\}$, $F = \{1, \cdots, m-1\}$ be Gambler's ruin.

Let $g(i) = \begin{cases} 1 & i = m \\ 0 & \text{otherwise} \end{cases}$ $\quad T =$ hitting time of $\{0, 1\}$.

$$h(k) = E_k\left[g(X_T)\right] = P_k(X_T = m)$$

$h(k) = \frac{1}{2}\left(h(k-1) + h(k+1)\right)$ since its harmonic.

$h(k) = Ck + d \qquad h(0) = d = 0 \qquad h(m) = Cm = 1 \Rightarrow C = \frac{1}{m}$

$h(k) = k/m \qquad$ (unique!)

Remark: Values of g only matter on $\partial F = \{y : \exists x \in F, p(x,y) > 0\}$

These are called discrete dirichlet problems.

Thm (13.38) Irreducible chains are transient iff $\exists$ a nonconstant superharmonic fn

Pf: If $h \geq 0$ be superharmonic. Assume for the sake of contradiction that the chain is recurrent

$\Rightarrow$ $h(X_n)$ is a supermartingale $\Rightarrow$ $h(X_n) \longrightarrow$ P_x a.s. Then this visits x & y infinitely many times, thus $h(x) = h(y)$ since $h(X_n)$ is convergent!

$\Rightarrow$ Chain is transient

Assume chain is transient. Let $h(y) = P_y(T_{\{x\}} < \infty)$

$h(y)$ is harmonic on $S \setminus \{x\}$ (we looked at $h(y) = E_y[g(X_T) 1_{\{T<\infty\}}]$)

$h(x) = 1$ obviously. $h(y) \leq 1$ $\forall y \in S \setminus \{x\}$ $\Rightarrow$ $Ph(x) = \sum p(x,y) h(y) \leq 1 = h(x)$

$\Rightarrow$ h is superharmonic on S.

If h is constant then $h(y) = 1 \Rightarrow P_y(T_{\{x\}} < \infty) = \rho_{yx} = 1$ Then

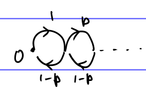
if $T_x = \inf\{n > 0 : X_n = x\}$ $P_x(T_x < \infty) = \sum p(x,y) P_y(T_x < \infty) + p(x,x)$

$$= \sum p(x,y) P_y(T_{\{x\}} < \infty) + p(x,x)$$

$$= \sum p(x,y) h(y) + p(x,x) = 1$$ $\Rightarrow$ $\rho_{xx} = 1$ and x is recurrent

Ex:

$$h(k) = \left(\frac{1-p}{p}\right)^k , \quad k \in \mathbb{Z}_+$$



is harmonic $h(0) = 1 \neq \left(\frac{1-p}{p}\right)^1$ So its not harmonic at 0.

But $h(i) = \left(\frac{1-p}{p}\right)^i = p\left(\frac{1-p}{p}\right)^{i+1} + (1-p)\left(\frac{1-p}{p}\right)^{i-1}$

$$= \left(\frac{1-p}{p}\right)^i (1-p+p) \checkmark$$

But $h(0) = 1 \geq Ph(0) = \frac{1-p}{p}$ when $p \geq 1/2$ For $p = 1/2$ $h(x) = 1$

So h is nonconstant superharmonic for $p > \frac{1}{2}$ $\Rightarrow$ transient

$\Rightarrow$ $p \leq \frac{1}{2}$ is recurrent.

Previously showed $\mu(k) = \left(\frac{p}{1-p}\right)^k$ was an invariant meas. Its finite for $p < 1/2$

$\Rightarrow$ the chain is +ve recurrent for $p < 1/2$ $\Rightarrow$ $p = 1/2$ is null recurrent.

Ex: Use this to show SRW is null recurrent on $\mathbb{Z}$.